# Unit : 01 [Ordinary Differential Equation of Higher order]

**Def$^n$ (Differential Equation) :** An equation involving the dependent variable, independent variable and the differential coefficient of the dependent variable with respect to the independent variable is known as a differential Equation.

- Differential Equation
  - Ordinary differential Equation (ODE)
  - Partial differential Equation (PDE)

**Def$^n$ (Ordinary Differential Equation):** A differential equation which involves only one independent variable is called an ordinary differential Eq$^n$

e.g $\boxed{\frac{d^2y}{dx^2} + y = \sin x}$

**Def$^n$ (Partial Differential Equation):** A differential equation which involves two or more independent variable and partial derivatives with respect to them is called Partial differential Equation.

e.g $\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z}$

**Def$^n$ (Order of the differential Equation):**
The order of a differential equation is the order of the higest ordered derivative occuring in the differential equation.

e.g. $\frac{dy}{dx} = \cot x$ [Order = 1], $\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^3 + y = 0$ [Order = 2]

**Def$^n$ (Degree of a Differential Equation):**
The degree of the differential equation is the degree of the higest ordered derivative present in the differential equation when it is made free from radicals signs and fractional powers.

e.g. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$ [Degree = 1]

$$\left(\frac{d^2y}{dx^2}\right) + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 0 \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3 \quad \text{[Degree = 2]}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{(d^2y/dx^2)} \Rightarrow \rho^2 = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3}{(d^2y/dx^2)^2} \quad \text{[Degree = 2]}$$

**General solution:** The general solution (or complete) solution of a differential equation is the solution in which the number of arbitrary constant is equal to the order of the differential equation.

e.g. $y = c_1 e^x + c_2 e^{-x}$ is the general sol$^n$ of the D.E. $y'' - y = 0$





Linear Differential equation of $n$th order with constant coefficient:

The equation of the form

$$\boxed{a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = Q} \quad (1)$$

Where $a_0, a_1, a_2, \dots, a_n$ are all constant and $Q$ is a function of $x$, is called Linear Diff$^n$ Equation of $n$th order with constant coefficient $(a_0 \neq 0)$

## Operator D:

Replace $\frac{d}{dx} \equiv D$, $\frac{d^2}{dx^2} \equiv D^2$, ---, $\frac{d^n}{dx^n} \equiv D^n$ in equ$^n$ (1)

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_{n-1} Dy + a_n y = Q$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = Q$$

$$\boxed{f(D)y = Q}$$

**Theorem:** If $y = u$ is the complete solution of the equation $f(D)y = 0$ and $y = v$ is a particular solution (containing no arbitrary constants) of the equ$^n$ $f(D)y = Q$, then the complete solution of the equ$^n$ $f(D)y = Q$ is $y = u + v$.

$$\boxed{y = C.F. + P.I}$$

$u$ = C.F. = complementary function
$v$ = P.I. = Particular integral.

## Steps for finding Auxiliary Equation.

**Step 1** Replace $y$ by $1$

**Step 2** Replace $\frac{dy}{dx}$ by $m$

**Step 3** Replace $\frac{d^2y}{dx^2}$ by $m^2$ and so on replace $\frac{d^ny}{dx^n}$ by $m^n$

**Step 4.** By doing so we get "Auxiliary Equation".

Let us consider a Second order linear differential eqn

$$(D^2+a_1D+a_2)y=0$$

where $a_1$, $a_2$ are constant.

The auxiliary Equation $m^2+a_1m+a_2=0$

If $m_1$, $m_2$ are the roots of given differential equation then

**{Case Ist}** If both $m_1$ and $m_2$ are real and distinct. then

$$\boxed{y(x)=C_1e^{m_1x}+C_2e^{m_2x}}$$

**{Case IInd}** If $m_1$ and $m_2$ are real & equal i.e $m_1=m_2$ then

$$\boxed{y(x)=(C_1+xC_2)e^{m_1x}}$$

**NOTE** If we have a IIIrd order differential eqn and it has three equal roots $m_1=m_2=m_3$ then

$$\boxed{y(x)=(C_1+xC_2+x^2C_3)e^{m_1x}}$$





**{Case III}** When $m_1$ and $m_2$ are complex roots.

Let $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$ then

$$\boxed{y(x)=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)}$$

**{Case IV}** When roots $m_1$ and $m_2$ are irrational.

Let $m_1=\alpha+\sqrt{\beta}$, $m_2=\alpha-\sqrt{\beta}$ then.

$$\boxed{y(x)=e^{\alpha x}(C_1\cosh\sqrt{\beta}x+C_2\sinh\sqrt{\beta}x)}$$

**NOTE** If we have IVth order linear differential Eqn

$$(D^4+a_1D^3+a_2D^2+a_3D+a_4)y=0$$

and it's auxiliary eqn $m^4+a_1m^3+a_2m^2+a_3m+a_4=0$

if roots $m_1=m_2=\alpha+i\beta$ and $m_3=m_4=\alpha-i\beta$ then

$$\boxed{y(x)=e^{\alpha x}\left[(C_1+xC_2)\cos\beta x+(C_3+xC_4)\sin\beta x\right]}$$

**Q1** $(D^2-3D+2)y=0$

**Ans** The auxiliary equation is given as

$$m^2-3m+2=0$$
$$m^2-2m-m+2=0$$
$$(m-2)(m-1)=0$$

$\Rightarrow m_1=1,\ m_2=2$

real and distinct

so Complementary function (C.F.)

$$\boxed{y_c(x)=C_1e^{x}+C_2e^{2x}}$$

Particular Integral (P.I.)

$$\boxed{y_p(x) = 0}$$

Complete solution.

$$y = C.F. + P.I. = c_1 e^{x} + c_2 e^{2x}.$$

Ans

**Q.2** Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

**Ans** $\Rightarrow (D^2 - 2D + 1)\, y = 0$

The auxiliary equ$^n$ is given as $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow \boxed{m = 1, 1}$

So $m_1 = m_2 = 1$ roots are real and equal.

(Complementary function) $y_c(x) = (c_1 + x c_2) e^{x}$

Particular integral (P.I.) $= 0$

Complete solution $y = C.F. + P.I.$

$$\boxed{y = (c_1 + x c_2) e^{x}}$$

**Q.3** Solve $\frac{d^2y}{dx^2} + 4y = 0$

$\Rightarrow (D^2 + 4)\, y = 0$

The auxiliary Equation $m^2 + 4 = 0 \Rightarrow \boxed{m = \pm 2i}$

Here $\alpha = 0$ and $\beta = 2$

So **C.F.** $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$

**P.I** $y_p(x) = 0$





Complete solution $y = C.F. + P.I.$

$$\boxed{y(x) = c_1 \cos(2x) + c_2 \sin(2x)}$$

**Q4:** Solve $(D^3 - 5D^2 + 8D - 4)\, y = 0$

**Auxiliary Equ$^n$** $m^3 - 5m^2 + 8m - 4 = 0$

$(m-1)(m^2 - 4m + 4) = 0$

$(m-1)(m-2)^2 = 0$

$$\boxed{m = 1, 2, 2}$$

$m_1 = 1,\ m_2 = m_3 = 2$

**C.F.** $y_c(x) = c_1 e^{x} + (c_2 + x c_3) e^{2x}$

**P.I.** $y_p(x) = 0$

Complete solution $y(x) = C.F. + P.I.$

$$\boxed{y(x) = c_1 e^{x} + (c_2 + x c_3) e^{2x}}$$

## Rules for finding the particular integral (P.I.)

Let $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n)\, y = Q$

It can be written as $f(D)\, y = Q$

$$\boxed{P.I. = \frac{1}{f(D)} Q}$$

**Case I** If $Q = e^{ax}$ i.e. $P.I. = \frac{1}{f(D)} e^{ax}$ then

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}, \text{ provided } f(a) \neq 0$$

**Case of failure** If $f(a)=0$ then

$$\frac{1}{f(D)}e^{ax} = x\cdot\frac{1}{f'(a)}e^{ax}, \quad \text{provided } f'(a)\neq 0$$

if $f'(a)=0$ then

$$\frac{1}{f(D)}e^{ax} = x\,x\,\frac{1}{f''(a)}e^{ax}, \quad \text{provided } f''(a)\neq 0$$

and so on...

$\boxed{\text{Case 2}}$ If $Q = \sin(ax+b)$ or $\cos(ax+b)$

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \quad \text{provided } f(-a^2)\neq 0$$

$$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \quad \text{provided } f(-a^2)\neq 0$$

**Case of failure** If $f(-a^2)=0$ then

$$\frac{1}{f(D^2)}\sin(ax+b) = x\,\frac{1}{f'(D^2)}\sin(ax+b), \quad \text{provided } f'(-a^2)\neq 0$$

If $f'(a^2)=0$, then

$$\frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{1}{f''(D^2)}\sin(ax+b), \quad \text{provided } f''(-a^2)\neq 0$$

and so on---

Q. Find the complete solution of the differential equation $(D-2)^3 y = 17e^{2x}$. [2011]

Ans The auxiliary equation is $(m-2)^3=0$





$m = 2,2,2$ $\quad m_1 = m_2 = m_3 = 2$ } Real and equal

C.F. $y_c(x) = (C_1 + xC_2 + x^2C_3)e^{2x}$

P.I. $y_p(x) = \frac{1}{(D-2)^3}17e^{2x} = 17\frac{1}{(D-2)^3}e^{2x}$

$$= 17x\left[\frac{1}{3(D-2)^2}e^{2x}\right]$$

$$= \frac{17}{3}x.x\frac{1}{2(D-2)}e^{2x}$$

$$= \frac{17}{6}x.x.x\,e^{2x} = \frac{17}{6}x^3e^{2x}$$

Complete solution $y(x) = C.F. + P.I.$

$$\boxed{y(x) = (C_1 + xC_2 + x^2C_3)e^{2x} + \frac{17}{6}x^3e^{2x}}$$

Q find the P.I. of $\frac{d^2y}{dx^2} + 4y = \sin 2x$ [2018-19]

Ans $(D^2+4)y = \sin 2x$

P.I. $\frac{1}{(D^2+4)}\sin 2x = x\cdot\frac{1}{2D}\sin 2x$

$$= \frac{x}{2}\int \sin 2x\,dx = -\frac{x}{2}\frac{\cos 2x}{2}$$

$$= -\frac{x}{4}\cos 2x$$

NOTE: $\frac{1}{D}Q = \int Q\,dx$

$\frac{1}{D-a}Q = e^{ax}\int Qe^{-ax}dx$

Q Solve $\frac{d^2y}{dx^2}+4y=\sin^2 2x$, with condition $y(0)=0$, $y'(0)=0$

Ans Auxiliary Equation $m^2+4=0$

$\boxed{m=\pm 2i}$

C.F. $y_c(x)=C_1\cos 2x+C_2\sin 2x$

P.I. $y_p(x)=\frac{1}{(D^2+4)}\sin^2 2x$

$=\frac{1}{(D^2+4)}\left(\frac{1-\cos 4x}{2}\right)$ $\left[\sin^2 x=\frac{1-\cos 2x}{2}\right]$

$=\frac{1}{2}\left[\frac{1}{D^2+4}-\frac{1}{D^2+4}\cos 4x\right]$

$=\frac{1}{2}\left[\frac{1}{D^2+4}e^{0x}-\frac{1}{D^2+4}\cos 4x\right]$

$=\frac{1}{2}\left[\frac{1}{0+4}-\frac{1}{-16+4}\cos 4x\right]$

$=\frac{1}{2}\left[\frac{1}{4}+\frac{1}{12}\cos 4x\right]$

Complete soln $y(x)=C.F.+P.I.$

$y(x)=C_1\cos 2x+C_2\sin 2x+\frac{1}{8}\left[1+\frac{\cos 4x}{3}\right]$

Using $y(0)=0$ we get

$0=C_1(1)+C_2(0)+\frac{1}{8}\left[1+\frac{1}{3}\right]$

$\Rightarrow \boxed{C_1=-1/6}$

$y'(x)=-2C_1\sin 2x+2C_2\cos 2x+\left[\frac{-\sin 4x}{6}\right]$

Using $y'(0)=0$ we get





$0=0+2C_2+0 \Rightarrow \boxed{C_2=0}$

so $y(x)=-\frac{1}{6}\cos 2x+\frac{1}{8}\left[1+\frac{\cos 4x}{3}\right]$

$\boxed{y(x)=\frac{\cos 4x}{24}-\frac{\cos 2x}{6}+\frac{1}{8}}$

NOTE If denominator reduces to a factor of the form $(\alpha D+\beta)$ then we operate by its conjugate $(\alpha D-\beta)$ on both numerator and denominator

Ques Find the P.I. of $(D^3+1)y=\sin(2x+1)$

Ans P.I. $=\frac{1}{(D^3+1)}\sin(2x+1)=\frac{1}{(DD^2+1)}\sin(2x+1)$

$=\frac{1}{D(-2^2)+1}\sin(2x+1)=\frac{1}{(-4D+1)}\sin(2x+1)$

Operating $(1+4D)$ in Drn and Nrn

$=\frac{(1+4D)}{(1+4D)}\left[\frac{1}{(1-4D)}\sin(2x+1)\right]$

$=\frac{(1+4D)(\sin(2x+1))}{1-16D^2}=\frac{(1+4D)}{1-16(-2^2)}\sin(2x+1)$

$=\frac{1}{65}\left[\sin(2x+1)+4D(\sin(2x+1))\right]$

$=\frac{1}{65}\left[\sin(2x+1)+8\cos(2x+1)\right]$

$\left(D=\frac{d}{dx}\right)$

**Case III** When $Q = x^m$, $m$ being a positive integer

$$\boxed{P.I. = \frac{1}{f(D)} x^m}$$

**Step 1** Take the lowest degree term common from $f(D)$ to get an expression of the form $[1 \pm \phi(D)]$ in the denominator and take it to numerator to become $[1 + \phi(D)]^{-1}$

**Step 2** Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to $n^{th}$ degree as $(n+1)^{th}$ derivative of $x^n$ is zero

**Step 3** Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$.

## Binomial Expansion.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$





**Q** $(D^2 - 3D + 2)y = x^2 + 2x + 1$ [2016]

**Ans** Auxiliary Equation $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$\boxed{m = 1, 2}$

Complementary function (C.F.)

$C.F. = y_c(x) = c_1 e^x + c_2 e^{2x}$

Particular integral (P.I.)

P.I. $\frac{1}{(D^2 - 3D + 2)}(x^2 + 2x + 1) = \frac{1}{2\left[1 + \frac{D^2 - 3D}{2}\right]}(x^2 + 2x + 1)$

$\Rightarrow \frac{1}{2}\left[1 + \frac{D^2 - 3D}{2}\right]^{-1}(x^2 + 2x + 1)$

$\Rightarrow \frac{1}{2}\left[1 - \left(\frac{D^2 - 3D}{2}\right) + \left(\frac{D^2 - 3D}{2}\right)^2\right](x^2 + 2x + 1)$ [Leaving higher power of D]

$\Rightarrow \frac{1}{2}\left[(x^2 + 2x + 1) - \frac{1}{2}(2 - 6x - 6) + \frac{1}{4}(9(2))\right]$

$\Rightarrow \frac{1}{2}\left[x^2 + 2x + 1 + 3x + 2 + \frac{9}{2}\right] = \frac{1}{2}\left[x^2 + 5x + \frac{15}{2}\right]$

Complete sol$^n$ $y(x) = C.F. + P.I.$

$$\boxed{y(x) = c_1 e^x + c_2 e^{2x} + \frac{1}{2}\left[x^2 + 5x + \frac{15}{2}\right]}$$

**Case IV** When $Q = e^{ax} V$

$$\boxed{P.I. = \frac{1}{f(D)}(e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V}$$

Q. $(D^2-2D+1)y = e^x \sin x$ [2016, 2017]

Ans. Auxiliary Equ$^n$ : $m^2-2m+1=0 \Rightarrow (m-1)^2=0$
$\Rightarrow \boxed{m=1,1}$

C.F. $y_c(x) = (c_1 + x c_2)e^x$

P.I. $\frac{1}{(D^2-2D+1)} e^x \sin x = \frac{1}{(D-1)^2} e^x \sin x$

$$= e^x \frac{1}{[(D+1)-1]^2} \sin x$$
$$= e^x \frac{1}{D^2} \sin x$$
$$= e^x \frac{1}{D} \int \sin x \, dx$$
$$= e^x \frac{1}{D}(-\cos x)$$
$$= -e^x \int \cos x \, dx$$
$$= -e^x \sin x$$

Complete solution $y(x) = C.F. + P.I.$

$$\boxed{y(x) = (c_1 + x c_2)e^x - e^x \sin x}$$

Q. $(D^2-2D+4)y = e^x \cos x + \sin x \cos(3x)$

Auxiliary Equ$^n$ $m^2-2m+4=0$

$$m = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}i$$

Complementary function (C.F.).

$$y_c(x) = e^x\left(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right)$$





P.I. $\frac{1}{(D^2-2D+4)}\left[e^x \cos x + \sin x \cos(3x)\right]$

$$\frac{1}{(D^2-2D+4)} e^x \cos x + \frac{1}{(D^2-2D+4)}[\sin x \cos(3x)]$$

$$\frac{1}{(D-1)^2+3} e^x \cos x + \frac{1}{2(D^2-2D+4)}[2\cos(3x)\sin x]$$

$$e^x \frac{1}{(D+1-1)^2+3} \cos x + \frac{1}{2(D^2-2D+4)}[\sin(4x) - \sin(2x)]$$

$$e^x \frac{1}{(D^2+3)} \cos x + \frac{1}{2(D^2-2D+4)} \sin(4x) - \frac{1}{2(D^2-2D+4)} \sin(2x)$$

$$e^x \frac{1}{(-1+3)} \cos x + \frac{1}{2(-16-2D+4)} \sin(4x) - \frac{1}{2(-4-2D+4)} \sin(2x)$$

$$\frac{e^x \cos x}{2} + \frac{1}{2(-12-2D)} \sin 4x - \frac{1}{2(-2D)} \sin 2x$$

$$\frac{e^x \cos x}{2} - \frac{1}{4}\frac{1}{(D+6)} \sin 4x + \frac{1}{4D} \sin(2x)$$

$$\frac{e^x \cos x}{2} - \frac{1}{4}\frac{(D-6)}{(D^2-36)} \sin(4x) + \frac{1}{4}\int \sin(2x)\,dx$$

$$\frac{e^x \cos x}{2} - \frac{1}{4}\frac{(D-6)}{(-16-36)} \sin(4x) + \frac{1}{8}(-\cos 2x)$$

$$\frac{e^x \cos x}{2} + \frac{1}{208}[4\cos 4x - 6\sin 4x] - \frac{\cos 2x}{8}$$

$$\frac{e^x \cos x}{2} - \frac{\cos 2x}{8} + \frac{1}{104}[2\cos 4x - 3\sin 4x]$$

Complete Sol$^n$ :: $y(x) = e^x[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x] + \frac{e^x \cos x}{2} - \frac{\cos 2x}{8} + \frac{1}{104}[2\cos 4x - 3\sin 4x]$

$\boxed{\text{Case V}}$ $\frac{1}{f(D)} x^n (\cos ax + i\sin ax) = \frac{1}{f(D)} x^n e^{iax}$

$(\because e^{iax} = \cos ax + i\sin ax)$

$$\frac{1}{f(D)} x^n (\cos ax + i\sin ax) = \frac{1}{f(D)} x^n e^{iax} = \frac{e^{iax}}{f(D+ia)} \cdot x^n$$

$$\frac{1}{f(D)} x^n \sin ax = \text{Imaginary part of } e^{iax} \frac{1}{f(D+ia)} x^n$$

$$\frac{1}{f(D)} x^n \cos ax = \text{Real part of } e^{iax} \frac{1}{f(D+ia)} x^n$$

**Q** Solve $(D^2+2D+1)y = x\cos x$

**Ans** Auxiliary Equation $m^2+2m+1=0$

$(m+1)^2 = 0$

$m = -1, -1$

C.F. $y_c(x) = (c_1 + x c_2) e^{-x}$

P.I. $y_p(x) = \frac{1}{(D^2+2D+1)} x\cos x = \frac{1}{(D+1)^2} x\cos x$

$= \text{Real part of } \frac{1}{(D+1)^2} x[\cos x + i\sin x]$

$= \text{Real part of } \frac{1}{(D+1)^2} x e^{ix}$

$= \text{Real part of } e^{ix} \frac{1}{(D+1+i)^2} x$

$= \text{Real part of } e^{ix} \frac{1}{D^2+(1+i)^2+2D(1+i)} x$

$= \text{R.P. of } e^{ix} \frac{1}{D^2+2i+2D(1+i)} x$

$= \text{R.P. of } \frac{e^{ix}}{2i}\left[1+\left(\frac{1+i}{i}\right)D+\frac{D^2}{2i}\right]^{-1} x$





$= \text{R.P. of } \frac{e^{ix}}{2i}\left[1-\left(\frac{1+i}{i}\right)D\right]x$ [leaving higher powers]

$= \text{R.P. of } \frac{e^{ix}}{2i}\left[x - \frac{1+i}{i}\right] = \text{R.P. of } \frac{e^{ix}}{(-2)}[xi-1-i]$

$= \text{R.P. of } \left(\frac{-1}{2}\right)(\cos x + i\sin x)(xi-1-i)$

$= \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$

The complete soln

$y(x) = y_c(x) + y_p(x)$

$$\boxed{y = (c_1 + x c_2)e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x}$$

**Q** Solve $(D^2-4D+4)y = 8x^2 e^{2x} \sin 2x$

**Ans** Auxiliary Equn $(m^2-4m+4)=0 \Rightarrow (m-2)^2=0$

$\Rightarrow m = 2, 2$

Complementary function (C.F.) $y_c(x) = (c_1 + x c_2)e^{2x}$

Particular integral (P.I.) $y_p(x) = \frac{1}{(D^2-4D+4)} 8x^2 e^{2x}\sin 2x$

$y_p(x) = \frac{1}{(D-2)^2}(8x^2 e^{2x}\sin 2x) = e^{2x}\frac{1}{(D+2-2)^2}(8x^2 \sin 2x)$

$= e^{2x}\frac{1}{D^2}(8x^2\sin 2x) = e^{2x}\frac{1}{D}\int 8x^2 \sin 2x\,dx$

$= 8e^{2x} \cdot \frac{1}{D}\left[-\frac{x^2}{2}\cos 2x - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right]$

$= 8e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]$

$$= 8e^{2x}\left[\int\left(\frac{-x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right)dx\right]$$

$$= 8e^{2x}\left[-\frac{x^2}{2}\left(\frac{\sin 2x}{2}\right) - \int\frac{(-2x)}{2}\left(\frac{\sin 2x}{2}\right)dx + \frac{x}{2}\left(-\frac{\cos 2x}{2}\right)\right.$$
$$\left. - \frac{1}{2}\int 1.\left(-\frac{\cos 2x}{2}\right) + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[-\frac{x^2}{4}\sin 2x + \frac{1}{2}\left\{x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right\}\right.$$
$$\left. - \frac{x}{4}\cos 2x + \frac{\sin 2x}{4} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[-\frac{x^2}{4}\sin(2x) + \frac{1}{8}\sin 2x - \frac{x}{4}\cos 2x + \frac{\sin 2x}{4}\right.$$
$$\left. - \frac{x}{4}\cos 2x + \frac{\sin 2x}{8} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(\frac{3}{8} - \frac{x^2}{4}\right)\sin(2x) - \frac{x}{2}\cos 2x\right]$$

Complete Soln $\quad y(x) = y_c(x) + y_p(x) = C.F. + P.I.$

$$y(x) = (C_1 + xC_2)e^{2x} + 8e^{2x}\left[\left(\frac{3}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x}{2}\cos 2x\right]$$

or $\boxed{y(x) = (C_1 + xC_2)e^{2x} + e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]}$

Q Solve the following diffn Equn [2012]

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 e^{-x}\cos x$$

Ans: $y = (C_1 + xC_2)e^{-x} + e^{-x}(-x^2\cos x + 4x\sin x + 6\cos x)$

(Home Work)





Case VI (General method of particular integral)

(i) $\frac{1}{D-\alpha}Q = e^{\alpha x}\int e^{-\alpha x}Q\,dx$

(ii) $\frac{1}{D+\alpha}Q = e^{-\alpha x}\int e^{\alpha x}Q\,dx$

Q Find the complete solution of $(D^2+a^2)y = \sec ax$ [2011, 2017]

Ans Auxiliary soln $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

C.F $= C_1\cos ax + C_2\sin ax$

P.I. $\frac{1}{(D^2+a^2)}\sec(ax) = \frac{1}{(D-ia)(D+ia)}\sec(ax)$

$$P.I = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax$$

(By partial fraction)

$$\Rightarrow \frac{1}{2ia}\left[\frac{1}{(D-ia)}\sec ax - \frac{1}{(D+ia)}\sec ax\right]$$

$$\Rightarrow \frac{1}{2ia}[P_1 - P_2] \quad \text{——— (1)}$$

$$P_1 = \frac{1}{D-ia}\sec(ax)$$

$$= e^{iax}\int e^{-iax}\sec(ax)dx$$

$$= e^{iax}\int(\cos ax - i\sin ax)\sec ax\,dx$$

$$= e^{iax}\int(1 - i\tan ax)dx$$

$$= e^{iax}\left[x + i\left(\frac{\log\cos ax}{a}\right)\right]$$

$$P_2 = \frac{1}{D+ia}\sec(ax)$$

$$= e^{-iax}\left[x - i\left(\frac{\log\cos ax}{a}\right)\right]$$

(Replacing $i$ by $-i$)

from (1)

$$P.I. = \frac{1}{2ia}\left[e^{iax}\left\{x + i\frac{\log\cos ax}{a}\right\} - e^{-iax}\left\{x - i\frac{\log\cos ax}{a}\right\}\right]$$

$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) + i\left(\frac{\log\cos ax}{a}\right)(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{a}\left[x\sin(ax) + \frac{1}{a}\cos ax\log(\cos ax)\right]$$

Complete soln $y = C.F + P.I.$

$$y = C_1\cos ax + C_2\sin ax + \frac{1}{a}\left[x\sin(ax) + \frac{1}{a}\cos ax\log(\cos ax)\right]$$

**Q** Find the general solution of the following differential Equ$^n$ $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

**Ans** Auxiliary Equ$^n$ $m^2 + 3m + 2 = 0$

$\Rightarrow (m+2)(m+1) = 0 \Rightarrow \boxed{m = -1, -2}$

C.F. $y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$

P.I. $y_p(x) = \frac{1}{(D^2+3D+2)} e^{e^x} = \frac{1}{(D+2)(D+1)} e^{e^x}$

$\Rightarrow \frac{1}{(D+2)(D+1)} e^{e^x}$

$\Rightarrow \left[\frac{1}{D+1} - \frac{1}{D+2}\right] e^{e^x}$

$\Rightarrow \left[\frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x}\right]$

$P_1 - P_2$

Now

$P_1 = \frac{1}{D+1} e^{e^x}$

$= e^{-x} \int e^x e^{e^x} dx$

[ Let $e^x = z$
then $e^x dx = dz$ ]

$= e^{-x} \int e^z dz$

$= e^{-x} e^z = e^{-x} e^{e^x}$

$P_2 = \frac{1}{D+2} e^{e^x}$

$= e^{-2x} \int e^{2x} e^{e^x} dx$

Let $e^x = u \Rightarrow e^x dx = du$

$= e^{-2x} \int u e^u du$

$= e^{-2x} [(u-1)e^u]$

$= e^{-2x} [(e^x - 1) e^{e^x}]$

$= (e^{-x} - e^{-2x}) e^{e^x}$

The complete sol$^n$ is given as –

$y(x) = y_c(x) + y_p(x)$

$y(x) = C_1 e^{-x} + C_2 e^{-2x} + e^{-x} e^{e^x} - (e^{-x} - e^{-2x}) e^{e^x}$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} e^{e^x}}$$





## Simultaneous Linear Differential Equ$^n$

Differential Equation in which there is one independent variable and two or more than two dependent variable. Such equation are called simultaneous linear differential equation.

e.g. $\frac{dx}{dt} + 4y = t$ and $\frac{dy}{dt} + 2x = e^t$

Here $x$ and $y$ are dependent variable and $t$ is independent variable.

The method of solving these equation is based on the process of elimination, as we solve algebraic simultaneous equations.

**Ques** Solve the following simultaneous differential equations: $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$ given that $x = y = 0$ when $t = 0$ [2015]

**Ans** Let $\frac{d}{dt} \equiv D$ then $\left(\frac{d}{dt} + 5\right)x - 2y = t$ and $\left(\frac{d}{dt} + 1\right)y + 2x = 0$ becomes.

$(D+5)x - 2y = t$ ——— (1)

$2x + (D+1)y = 0$ ——— (2)

Operating (1) by $(D+1)$ and multiply (2) by 2 and add

$(D+1)(D+5)x - 2(D+1)y = (D+1)t$ ——— (3)

$4x + 2(D+1)y = 0$ ——— (4)

Adding (3) and (4)

$$(D^2+6D+5)x+4x=(D+1)t$$

or $$(D^2+6D+9)x=1+t$$

Auxiliary Equ^n

$$m^2+6m+9=0 \Rightarrow (m+3)^2=0 \Rightarrow \boxed{m=-3,-3}$$

C.F. $= (C_1+tC_2)e^{-3t}$

P.I. $= \frac{1}{(D+3)^2}(1+t) = \frac{1}{9}\left(1+\frac{D}{3}\right)^{-2}(1+t)$

$$= \frac{1}{9}\left(1-\frac{2D}{3}\right)(1+t) = \frac{1}{9}\left(1+t-\frac{2}{3}\right) = \frac{1}{9}\left(t+\frac{1}{3}\right)$$

So $x = C.F.+P.I. \Rightarrow \boxed{x=(C_1+tC_2)e^{-3t}+\frac{1}{9}\left(t+\frac{1}{3}\right)}$ —— (5)

Now $\frac{dx}{dt} = (C_1+tC_2)e^{-3t}(-3)+C_2e^{-3t}+\frac{1}{9}(1+0)$

$$\frac{dx}{dt} = -3(C_1+tC_2)e^{-3t}+C_2e^{-3t}+\frac{1}{9} \quad \text{—— (6)}$$

put the value of $x$ and $\frac{dx}{dt}$ in equ^n $\frac{dx}{dt}+5x-2y=t$ we get

$$-3(C_1+tC_2)e^{-3t}+C_2e^{-3t}+\frac{1}{9}+5\left[(C_1+tC_2)e^{-3t}+\frac{1}{9}\left(t+\frac{1}{3}\right)\right] -2y=t$$

$$\Rightarrow \quad 2y = -3(C_1+tC_2)e^{-3t}+C_2e^{-3t}+\frac{1}{9}+5(C_1+tC_2)e^{-3t}+\frac{5}{9}\left(t+\frac{1}{3}\right)-t$$

$$\boxed{y=(C_1+tC_2)e^{-3t}+\frac{C_2}{2}e^{-3t}-\frac{2}{9}t+\frac{4}{27}} \quad \text{—— (7)}$$





Using $x(0)=0$ i.e. $x=0$ when $t=0$ in equ^n (5)

$$0=(C_1+0)+\frac{1}{9}\left(0+\frac{1}{3}\right) \Rightarrow \boxed{C_1=-1/27}$$

Using $y(0)=0$ i.e. $y=0$ when $t=0$ in equ^n (7)

$$0=(C_1+0)+\frac{C_2}{2}+\frac{4}{27} \Rightarrow \boxed{C_2=-2/9}$$

Putting the value of $C_1$ and $C_2$ in (5) and (7), we get

$$x=\left(\frac{-1}{27}-\frac{2}{9}t\right)e^{-3t}+\frac{1}{9}\left(t+\frac{1}{3}\right) \Rightarrow \boxed{x=\frac{-1}{27}[1+6t]e^{-3t}+\frac{1}{9}\left[t+\frac{1}{3}\right]}$$

$$y=\left(\frac{-1}{27}-\frac{2}{9}t\right)e^{-3t}+\frac{1}{2}\left(\frac{-2}{9}\right)e^{-3t}-\frac{2}{9}t+\frac{4}{27}$$

$$\boxed{y=\frac{-2}{27}(2+3t)e^{-3t}-\frac{2}{9}t+\frac{4}{27}}$$

Ques Solve $\frac{dx}{dt}=-4(x+y)$ and $\frac{dx}{dt}+4\frac{dy}{dt}=-4y$ with conditions $x(0)=1$, $y(0)=0$ [2014, 2011]

Ans Let $\frac{d}{dt}\equiv D$ then

$$(D+4)x+4y=0 \quad \text{—— (1)}$$

$$Dx+4(D+1)y=0 \quad \text{—— (2)}$$

operating $(D+1)$ in (1) and subtracting from (2)

$$(D+1)(D+4)x+4(D+1)y=0$$

$$Dx+4(D+1)y=0$$

$$(D^2+5D+4)x-Dx=0 \Rightarrow (D^2+4D+4)x=0$$

$$\Rightarrow (D+2)^2x=0$$

Auxiliary Equ^n $(m+2)^2=0 \Rightarrow \boxed{m=-2,-2}$

C.F. $=(C_1+tC_2)e^{-2t}$ $\quad\boxed{P.I.=0}$

So $\boxed{x=(C_1+tC_2)e^{-2t}}$

$$\frac{dx}{dt} = (c_1 + tc_2)e^{-2t}(-2) + c_2 e^{-2t} = (-2c_1 + c_2 - 2tc_2)e^{-2t}$$

Now from ①

$$y = -\frac{1}{4}\left[\frac{dx}{dt} - 4x\right]$$

$$y = -\frac{1}{4}\left[(-2c_1 + c_2 - 2tc_2)e^{-2t} + 4(c_1 + tc_2)e^{-2t}\right]$$

$$\boxed{y = -\frac{1}{4}\left[2c_1 + c_2 + 2tc_2\right]e^{-2t}}$$

using $x(0) = 1$, we get

$1 = (c_1 + 0) \Rightarrow \boxed{c_1 = 1}$

using $y(0) = 0$, we get

$0 = -\frac{1}{4}[2 + c_2] \Rightarrow \boxed{c_2 = -2}$

So

$$\boxed{\begin{aligned} x &= (1-2t)e^{-2t} \\ y &= te^{-2t} \end{aligned}}$$

**Que:** Solve the following

$\frac{dx}{dt} = 3x + 8y$ ; $\frac{dy}{dt} = -x - 3y$, with $x(0) = 6$ and $y(0) = -2$

(Home work)

**Que:** Solve the simultaneous eqn

$$x'(t) = y, \quad y'(t) = -x, \quad x(0) = 0,\ y(0) = 0$$

(Home work)





## Simultaneous Differential Equation of Second Order

**Que:** Solve the simultaneous differential equations

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y \quad \text{——①} \quad \text{and} \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t \quad \text{——②}$$ [2018]

**Ans** Let $\frac{d}{dt} \equiv D$, $\frac{d^2}{dt^2} \equiv D^2$ then equation ① and ② becomes.

$$(D^2 - 4D + 4)x - y = 0 \quad \text{——③}$$

$$-25x + (D^2 + 4D + 4)y = 16e^t \quad \text{——④}$$

operating ③ by $(D^2 + 4D + 4)$ and adding to eqn ④

$$(D^2 + 4D + 4)(D^2 - 4D + 4)x - (D^2 + 4D + 4)y = 0$$

$$-25x + (D^2 + 4D + 4)y = 16e^t$$

$$-25x + (D^2 + 4D + 4)(D^2 - 4D + 4)x = 16e^t$$

$$-25x + [(D^2+4)^2 - 16D^2]x = 16e^t$$

$$(D^4 - 8D^2 - 9)x = 16e^t$$

Auxiliary Equation: $m^4 - 8m^2 - 9 = 0$

$m^4 - 9m^2 + m^2 - 9 = 0$

$\Rightarrow (m^2 - 9)(m^2 + 1) = 0 \Rightarrow \boxed{m = \pm i, \pm 3}$

$$\text{C.F.} = c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t$$

$$\text{P.I.} = \frac{1}{(D^4 - 8D^2 - 9)}(16e^t) = 16\frac{1}{(D^4 - 8D^2 - 9)}e^t = \frac{16e^t}{(1-8-9)}$$

$$= -e^t$$

so $x = C.F. + P.I. \Rightarrow \boxed{x = c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t - e^{t}}$

$\frac{dx}{dt} = 3c_1 e^{3t} - 3c_2 e^{-3t} - c_3 \sin t + c_4 \cos t - e^{t}$

$\frac{d^2x}{dt^2} = 9c_1 e^{3t} + 9c_2 e^{-3t} - c_3 \cos t - c_4 \sin t - e^{t}$

Now put the value of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$ in ①

$(9c_1 e^{3t} + 9c_2 e^{-3t} - c_3 \cos t - c_4 \sin t - e^{t})$
$- 4(3c_1 e^{3t} - 3c_2 e^{-3t} - c_3 \sin t + c_4 \cos t - e^{t})$
$+ 4(c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t - e^{t}) = y$

so $\boxed{y = c_1 e^{3t} + 25 e^{-3t} + (3c_3 - 4c_4)\cos t + (4c_3 + 3c_4)\sin t - e^{t}}$

Ques$^n$ Solve the simultaneous equations:

$\frac{d^2x}{dt^2} + y = \sin t$ ——① and $\frac{d^2y}{dt^2} + x = \cos t$ ——②

Ans Let $\frac{d}{dt} \equiv D$ and $\frac{d^2}{dt^2} \equiv D^2$ then from ① and ②

$D^2 x + y = \sin t$ ——— ③

$x + D^2 y = \cos t$ ——— ④

Operating ① by $D^2$ and subtract from (4)

$$\begin{array}{r} D^4 x + D^2 y = D^2(\sin t) \\ x + D^2 y = \cos t \\ \hline (D^4 - 1)x = -\sin t - \cos t \end{array}$$





Auxiliary Equ$^n$ $m^4 - 1 = 0 \Rightarrow (m^2-1)(m^2+1) = 0$
$\Rightarrow \boxed{m = \pm 1, \pm i}$

C.F. $= c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

P.I. $= \frac{1}{(D^4-1)}(-\sin t - \cos t) = \frac{-1}{(D^4-1)}(\sin t + \cos t)$

$= (-t)\left[\frac{1}{4D^3}(\sin t + \cos t)\right]$ $\left[\frac{1}{f(D^2)}\sin(ax) = x \cdot \frac{1}{f'(D^2)}\sin(ax) \text{ if } f(-a^2) = 0\right]$

$= \frac{-1}{4} t\,[\cos t - \sin t]$

so $x = C.F. + P.I. \Rightarrow \boxed{x = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t)}$

Now

$\frac{dx}{dt} = c_1 e^{t} - c_2 e^{-t} - c_3 \sin t + c_4 \cos t + \frac{1}{4}(\sin t - \cos t) + \frac{t}{4}(\cos t + \sin t)$

$\frac{d^2x}{dt^2} = c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t + \frac{1}{4}[\cos t + \sin t] + \frac{1}{4}[\cos t + \sin t] + \frac{t}{4}[-\sin t + \cos t]$

$\frac{d^2x}{dt^2} = c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t + \frac{1}{2}[\cos t + \sin t] + \frac{t}{4}[-\sin t + \cos t]$

from ①

$c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t + \frac{1}{2}(\cos t + \sin t) + \frac{t}{4}(-\sin t + \cos t) = \sin t - y$

so $\boxed{y = -c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}[\sin t - \cos t] + \frac{1}{2}[\sin t - \cos t]}$

So $x = C.F. + P.I. \Rightarrow \boxed{x = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t}$

$$\frac{dx}{dt} = 3C_1 e^{3t} - 3C_2 e^{-3t} - C_3 \sin t + C_4 \cos t - e^t$$

$$\frac{d^2x}{dt^2} = 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t$$

Now put the value of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$ in ①

$$(9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t)$$
$$- 4(3C_1 e^{3t} - 3C_2 e^{-3t} - C_3 \sin t + C_4 \cos t - e^t)$$
$$+ 4(C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t) = y$$

So $\boxed{y = C_1 e^{3t} + 25 e^{-3t} + (3C_3 - 4C_4) \cos t + (4C_3 + 3C_4) \sin t - e^t}$

**Ques$^n$** Solve the simultaneous equations:

$$\frac{d^2x}{dt^2} + y = \sin t \quad ① \qquad \text{and} \qquad \frac{d^2y}{dt^2} + x = \cos t \quad ②$$

**Ans** Let $\frac{d}{dt} \equiv D$ and $\frac{d^2}{dt^2} \equiv D^2$ then from ① and ②

$$D^2 x + y = \sin t \quad ③$$
$$x + D^2 y = \cos t \quad ④$$

Operating ① by $D^2$ and subtract from (4)

$$D^4 x + D^2 y = D^2(\sin t)$$
$$x + D^2 y = \cos t$$
$$\overline{(D^4 - 1)x = -\sin t - \cos t}$$





Auxiliary Equ$^n$ $m^4 - 1 = 0 \Rightarrow (m^2-1)(m^2+1) = 0$
$\Rightarrow \boxed{m = \pm 1, \pm i}$

C.F. $= C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$

P.I. $= \frac{1}{(D^4-1)}(-\sin t - \cos t) = \frac{-1}{(D^4-1)}(\sin t + \cos t)$

$= (-t)\left[\frac{1}{4D^3}(\sin t + \cos t)\right]$ $\left[\frac{1}{f(D^2)} \sin(ax) = x \cdot \frac{1}{f'(D^2)} \sin(ax) \text{ if } f(-a^2) = 0\right]$

$= \frac{-1}{4} t[\cos t - \sin t]$

So $x = C.F. + P.I. \Rightarrow \boxed{x = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4}(\sin t - \cos t)}$

Now

$$\frac{dx}{dt} = C_1 e^t - C_2 e^{-t} - C_3 \sin t + C_4 \cos t + \frac{1}{4}(\sin t - \cos t) + \frac{t}{4}(\cos t + \sin t)$$

$$\frac{d^2x}{dt^2} = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t + \frac{1}{4}[\cos t + \sin t] + \frac{1}{4}[\cos t + \sin t] + \frac{t}{4}[-\sin t + \cos t]$$

$$\frac{d^2x}{dt^2} = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t + \frac{1}{2}[\cos t + \sin t] + \frac{t}{4}[-\sin t + \cos t]$$

from ①

$$C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t + \frac{1}{2}(\cos t + \sin t) + \frac{t}{4}(-\sin t + \cos t) = \sin t - y$$

So $\boxed{y = -C_1 e^t - C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4}[\sin t - \cos t] + \frac{1}{2}[\sin t - \cos t]}$

# Homogeneous Linear Differential Equation

## (Euler- Cauchy Equations)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x\frac{dy}{dx} + a_n y = Q$$

where $a_1, a_2, \ldots, a_n$ are constants and $Q$ is a function of $x$, is called "Cauchy's homogeneous linear Equations."

To solve this problem we use substitution.

$$x = e^z \quad \text{or} \quad z = \log x$$

**Steps for solution**

1) Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$

2) Replace $x\frac{d}{dx}$ by $D$

$x^2\frac{d^2}{dx^2}$ by $D(D-1)$

$x^3\frac{d^3}{dx^3}$ by $D(D-1)(D-2)$ and so on...

3) By doing so, this type of equation reduces to linear differential equation with constant coeff. Which is then solved as before.





**Que** Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$ — ①

Putting $x = e^z$, $D \equiv \frac{d}{dz}$, $x\frac{d}{dx} \equiv D^2$, $x^2\frac{d}{dx^2} \equiv D(D-1)$

so from ①

$$D(D-1)y + Dy + y = \log(e^z)\sin(\log e^z)$$

$$(D(D-1) + D + 1)y = z\sin z$$

$$(D^2+1)y = z\sin z$$

Auxiliary eqn $m^2+1=0 \Rightarrow \boxed{m = \pm i}$

C.F. $= c_1\cos z + c_2 \sin z$

P.I. $= \frac{1}{(D^2+1)} z\sin z$

$= \text{Imag part of } \frac{1}{(D^2+1)} z e^{iz}$

$= \text{Imag part of } e^{iz}\frac{1}{(D+i)^2+1} z$

$= \text{Imag. part of } e^{iz}\frac{1}{(D^2+2Di)} z$

$= \text{Imag. part of } \frac{e^{iz}}{2Di}\left(1+\frac{D}{2i}\right)^{-1} z$

$= \text{Imag. part of } \frac{e^{iz}}{2Di}\left(1-\frac{D}{2i}\right) z$

$= \text{Imag part of } \frac{e^{iz}}{2Di}\left(z - \frac{1}{2i}\right)$

$= \text{Imag. part of } e^{iz}\left(\frac{z^2}{4i} + \frac{z}{4}\right)$

$= \text{Imag. part of } (\cos z + i\sin z)\left(-\frac{z^2 i}{4} + \frac{z}{4}\right)$

$= -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z = \frac{z}{4}\left[\sin z - z\cos z\right]$

So $y = C.F. + P.I.$

$$y = c_1 \cos z + c_2 \sin z + \frac{z}{4}[\sin z - z\cos z]$$

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) + \frac{\log x}{4}[\sin(\log x) - \log x \cos(\log x)]$$

$$\boxed{y = c_1 \cos(\log x) + c_2 \sin(\log x) + \frac{\log x}{4}[\sin(\log x) - \log x \cos(\log x)]}$$

**Ques** Solve $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$ [2011]

**Ans** Putting $x+1 = e^z \Rightarrow z = \log(x+1)$ and let

$D \equiv \frac{d}{dz}$, $\frac{d^2}{dz^2} \equiv D(D-1)$ so form given equ$^n$

$$[D(D-1) + D]y = (2e^z+1)(2e^z+2)$$

$$D^2y = 4e^{2z} + 6e^z + 2$$

$$\left[\begin{array}{l} 2x+3 = 2(x+1)+1 \\ \quad = 2e^z+1 \\ \text{(and)} \\ 2x+4 = 2(x+1)+2 \\ \quad = 2e^z+2 \end{array}\right]$$

Auxiliary Equ$^n$ $m^2 = 0 \Rightarrow \boxed{m = 0, 0}$

C.F. $(c_1 + z c_2)$

P.I. $\frac{1}{D^2}[4e^{2z} + 6e^z + 2] = 4\left(\frac{e^{2z}}{4}\right) + 6e^z + 2\left(\frac{z^2}{2}\right)$

$$= e^{2z} + 6e^z + z^2$$

Complete sol$^n$

$$y = C.F. + P.I.$$

$$y = c_1 + z c_2 + e^{2z} + 6e^z + z^2$$

$$\boxed{y = c_1 + \log(x+1)\, c_2 + (x+1)^2 + 6(x+1) + [\log(x+1)]^2}$$





## Method of Reduction of Order

Consider the standard form of second order linear differential equation with variable coefficient.

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R} \quad \text{--- (1)}$$

Where P, Q and R are function of $x$ alone.

**Steps for Solution by Method of Reduction of Order**

- Compare the given differential equation with equ$^n$ (1) and find P, Q and R.
- Apply the below mentioned condition if any one of the condition is satisfied, write down the part of complementary function (part of C.F.) as $u$.

> If $1 + P + Q = 0$ then part of C.F. $= e^x$
> If $1 - P + Q = 0$ then part of C.F. $= e^{-x}$
> If $m^2 + mP + Q = 0$ then part of C.F. $= e^{mx}$
> If $P + Qx = 0$ then part of C.F. $= x$
> If $2 + 2Px + Qx^2 = 0$ then part of C.F. $= x^2$
> If $n(n-1) + Pnx + Qx^2 = 0$ then part of C.F. $= x^n$

- Let $y = uv$ is the complete solution of the given differential equation.

- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and substitute these values in differential equation.
- Put $\frac{dv}{dx} = z$ and $\frac{d^2v}{dx^2} = \frac{dz}{dx}$ then we get a first order differential equation.
- Solve the first order differential equation to find $v$.

$\Rightarrow$ $\boxed{y = vu \text{ is complete solution.}}$

**Ques** Solve $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$

**Ans**

$$\frac{d^2y}{dx^2} - \frac{2x(1+x)}{x^2}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x$$

$$\frac{d^2y}{dx^2} - 2\left(\frac{1}{x}+1\right)\frac{dy}{dx} + 2\left(\frac{1}{x^2}+\frac{1}{x}\right)y = x \quad \text{—— (1)}$$

Comparing differential equ$^n$ (1) with

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ we get

$$P = -2\left(\frac{1}{x}+1\right) \quad \text{and} \quad Q = 2\left(\frac{1}{x^2}+\frac{1}{x}\right)$$

$$P + Qx = -2\left(\frac{1}{x}+1\right) + 2x\left(\frac{1}{x^2}+\frac{1}{x}\right) = 0$$

Hence $y = x$ is a part of C.F. so $\boxed{u = x}$

Let $y = uv = xv$ is a complete solution of the given differential equation.

$$y = vx$$





$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Put in (1)

$$\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2\left(\frac{1}{x}+1\right)\left(v + x\frac{dv}{dx}\right) + 2\left(\frac{1}{x^2}+\frac{1}{x}\right)vx = x$$

$$x\frac{d^2v}{dx^2} + (2-2-2x)\frac{dv}{dx} + \left(\frac{-2}{x} - 2 + \frac{2}{x} + 2\right)v = x$$

$$x\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} = x$$

$$\boxed{\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1}$$

Let $\frac{dv}{dx} = z$ and $\frac{d^2v}{dx^2} = \frac{dz}{dx}$

$\boxed{\frac{dz}{dx} - 2z = 1} \longrightarrow$ This is a first order linear differential Equ$^n$

Integrating factor $(I.F.) = e^{\int -2dx} = e^{-2x}$

Solution $z(I.F.) = \int (I.F.)(1)\,dx + C_1$

$$ze^{-2x} = \int e^{-2x}dx + C_1$$

$$ze^{-2x} = \frac{e^{-2x}}{(-2)} + C_1$$

$$\boxed{z = \left(\frac{-1}{2}\right) + C_1e^{2x}}$$

put $z = \frac{dv}{dx}$

$$\frac{dv}{dx} = \left(\frac{-1}{2}\right) + C_1e^{2x}$$

$$dv = \left(\left(\frac{-1}{2}\right) + c_1 e^{2x}\right) dx$$

Integrating both sides, we get

$$\boxed{v = \frac{-x}{2} + \frac{c_1}{2} e^{2x} + c_2}$$

Hence complete solution is given as

$$y = uv \Rightarrow \boxed{y = \left(\frac{-x}{2} + \frac{c_1}{2} e^{2x} + c_2\right) x}$$

NOTE [Solution of first order linear differential Equation]

$\frac{dy}{dx} + Py = Q$ then solution. [P and Q are funcⁿ of $x$]

$y(I.F.) = \int (I.F.) Q\, dx + C$ where $I.F. = e^{\int P dx}$

**Ques** Solve $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x) y = e^x \sin x$

**Ans** $\boxed{y = \left[\frac{-1}{2} \cos x - \frac{1}{5} c_1 e^{-2x} (\cos x + 2 \sin x) + c_2\right] e^x}$

(Hint Check $1 + P + Q = 0$ ?)





## Normal Form Method [Removal of first Derivative]

When the part of C.F. can not be determined by the method of Reduction of order, we reduce the given differential equation in Normal form

**Steps for solutions**

- Make the coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so
- Compare the given differential equⁿ with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ and find P, Q and R.
- Let $y = uv$ be complete solution
- Find $\boxed{u = e^{-\frac{1}{2}\int P dx}}$

$$\boxed{I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}} \text{ and } \boxed{S = \frac{R}{u}}$$

- Check $I$ is a constant or $\frac{\text{constant}}{x^2}$ ? If not method is not applicable.
- Case (i) if $I$ is constant
  ⇒ We get a Second order linear differential equⁿ with constant coefficient.

  Case (ii) if $I$ is $\frac{\text{constant}}{x^2}$
  ⇒ We get a homogeneous linear diffⁿ Equⁿ with variable coefficient.

•> Now Normal form is given by $\frac{d^2v}{dx^2}+Iv=S$ which we solve for $v$

•> $\boxed{y=uv}$ be complete solution of the given differential Equn.

Ques Solve $x^2\frac{d^2y}{dx^2}-2(x^2+x)\frac{dy}{dx}+(x^2+2x+2)y=0$ by normal form. [2019]

Ans $x^2\frac{d^2y}{dx^2}-2(x^2+x)\frac{dy}{dx}+(x^2+2x+2)y=0$

$$\frac{d^2y}{dx^2}-2\left(\frac{x^2+x}{x^2}\right)\frac{dy}{dx}+\left(\frac{x^2+2x+2}{x^2}\right)y=0$$

$$\frac{d^2y}{dx^2}-2\left(1+\frac{1}{x}\right)\frac{dy}{dx}+\left(1+\frac{2}{x}+\frac{2}{x^2}\right)y=0 \quad \text{——} ①$$

On comparing Equn ① with $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, we get

$$P=-2\left(1+\frac{1}{x}\right),\quad Q=1+\frac{2}{x}+\frac{2}{x^2},\quad R=0$$

Let $y=uv$ be the complete solution of the given differential equation.

Now $u=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}\int -2(1+\frac{1}{x})dx}$

$=e^{\int(1+\frac{1}{x})dx}=e^{x+\ln x}=e^{x}e^{\ln x}$

$\boxed{u=xe^{x}}$





$I=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2=\left(1+\frac{2}{x}+\frac{2}{x^2}\right)-\frac{1}{2}\left(\frac{2}{x^2}\right)-\frac{1}{4}\left(-2\left(1+\frac{1}{x}\right)\right)^2$

$=1+\frac{2}{x}+\frac{2}{x^2}-\frac{1}{x^2}-1-\frac{1}{x^2}-\frac{2}{x}=0$

$\boxed{I=0}$

$S=\frac{R}{u} \Rightarrow \boxed{S=0}$

Now Normal form is

$$\frac{d^2v}{dx^2}+Iv=S$$

$$\boxed{\frac{d^2v}{dx^2}=0}$$

Auxiliary Equn $m^2=0 \Rightarrow \boxed{m=0,0}$

C.F. $v_c=(c_1+xc_2)$

P.I $v_p=0$

Complete soln $\boxed{v=c_1+xc_2}$

Now the complete solution of differential Equn ① is given as $y=uv$

$\boxed{y=xe^{x}(c_1+xc_2)}$

Home work

Ques Solve $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$.

Ans: $\boxed{y=e^{x^2}(c_1\cos x+c_2\sin x+\sin 2x)}$

**Ques** Solve $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \left[\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right]y = 0$

**Ans** On comparing given differential equation with

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ we get

$P = x^{-1/3}$, $Q = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}$, $R = 0$

Let $y = uv$ be the complete solution of given eq$^n$

Now $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int x^{-1/3}} = e^{-\frac{1}{2}\left(\frac{x^{2/3}}{2/3}\right)}$

$\boxed{u = e^{-\frac{3}{4}x^{2/3}}}$

$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = \frac{1}{4}x^{-2/3} - \frac{1}{6}x^{-4/3} - \frac{6}{x^2} - \frac{1}{2}\left[\frac{-1}{3}x^{-4/3}\right] - \frac{1}{4}x^{-2/3}$

$I = -\frac{6}{x^2}$ $\left(\frac{\text{constant}}{x^2}\right)$

$S = \frac{R}{u} = 0$

Now Normal form is. $\frac{d^2v}{dx^2} + Iv = S$

$\frac{d^2v}{dx^2} - \frac{6}{x^2}v = 0 \Rightarrow x^2\frac{d^2v}{dx^2} - 6v = 0$

Let $x = e^z \Rightarrow z = \log_e x$  Let $D \equiv \frac{d}{dz}$

$D(D-1) \equiv \frac{d^2}{dz^2}$

$[D(D-1) - 6]v = 0 \Rightarrow (D^2 - D - 6)v = 0$





Auxiliary Eq$^n$ $m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0$

$\Rightarrow \boxed{m = -2, 3}$

$C.F. = c_1 e^{-2z} + c_2 e^{3z}$

$P.I. = 0$

So $v = C.F. + P.I.$

$v = c_1 e^{-2z} + c_2 e^{3z}$

$\boxed{v = c_1 x^{-2} + c_2 x^3}$

$\Rightarrow$ Complete sol$^n$

$y = uv$

$\boxed{y = e^{-\frac{3}{4}x^{2/3}}\left[c_1 x^3 + \frac{c_2}{x^2}\right]}$

Home Work

**Ques** $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x \sec x$ [2015]

**Ans** $y = \sec x\left[c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x + \frac{e^x}{4}\right]$

**Ques** $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2-8)y = x^2 e^{-x^2/2}$ — (1) [2012, 2013]

**Ans** On comparing equ$^n$ (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 2x$, $Q = x^2 - 8$, $R = x^2 e^{-x^2/2}$

Let $y = uv$ be the complete solution of given eq

Now

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int 2x\,dx} = e^{-x^2/2}$$

$$\boxed{u = e^{-x^2/2}}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = (x^2-8) - \frac{1}{2}(2) - \frac{1}{4}(4x^2) = -9$$

$$\boxed{I = -9}$$

$$S = \frac{R}{u} = \frac{x^2 e^{-x^2/2}}{e^{-x^2/2}} \Rightarrow \boxed{S = x^2}$$

So Normal form is $\frac{d^2v}{dx^2} + Iv = S$

$$\frac{d^2v}{dx^2} - 9v = x^2$$

Auxiliary Eqn $m^2 - 9 = 0 \Rightarrow \boxed{m = \pm 3}$

C.F $= c_1 e^{-3x} + c_2 e^{3x}$

P.I $= \frac{1}{D^2-9}x^2 = \frac{-1}{9}\left[1 - \frac{D^2}{9}\right]^{-1} x^2 = \frac{-1}{9}\left(x^2 + \frac{2}{9}\right)$

So $v = c_1 e^{-3x} + c_2 e^{3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)$

Hence complete soln is given as

$$y = uv$$

$$\boxed{y = e^{-x^2/2}\left[c_1 e^{-3x} + c_2 e^{3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)\right]}$$





## Change of independent variable method

Consider the linear differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{———} (1)$$

We change the independent variable from $x$ to $z$.
where $z = f(x)$ i.e. $z$ is a function of $x$

then Eqn (1) becomes

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{———} (2)$$

Where

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

### Steps for Soln

- Make the coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not.
- Compare given differential Eqn with (1)
- Find $P$, $Q$ and $R$.
- Choose $z$ such that $\left(\frac{dz}{dx}\right)^2 = Q$

Here $Q$ is taken in such a way that it remain the whole square of a function without surd and its negative sign is ignored

→ find $\frac{dz}{dx}$, $z$ and $\frac{d^2z}{dx^2}$

→ $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

→ Solve $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

→ Now we find the sol$^n$ of $y$ in terms of $x$ by replacing $z$ by $x$.

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**Ques** Solve by the changing the independent variable
$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$ [2015]

**Ans**
$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x \quad \text{—— (1)}$$

On comparing equ$^n$ (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 3\sin x - \cot x$, $Q = 2\sin^2 x$, $R = e^{-\cos x}\sin^2 x$

Choose $z$ such that
$$\left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \frac{dz}{dx} = \sin x$$
$$\Rightarrow z = -\cos x$$
and $\frac{d^2z}{dx^2} = \cos x$

Now
$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x + (3\sin x - \cot x)\sin x}{(\sin^2 x)}$$





$$P_1 = \frac{\cos x + 3\sin^2 x - \cos x}{\sin^2 x} \Rightarrow \boxed{P_1 = 3}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{2\sin^2 x}{\sin^2 x} = 2 \Rightarrow \boxed{Q_1 = 2}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{e^{-\cos x}\sin^2 x}{\sin^2 x} \Rightarrow \boxed{R_1 = e^{-\cos x}}$$

Hence
$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{-\cos x} = e^z$$
i.e. $$\boxed{\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^z}$$

Auxiliary equ$^n$ $m^2 + 3m + 2 = 0$
$\Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -1, -2$

C.F. $= C_1 e^{-z} + C_2 e^{-2z}$

P.I. $= \dfrac{1}{(D^2+3D+2)} e^z = \dfrac{1}{1+3+2} e^z = \dfrac{1}{6} e^z$

so $y = \text{C.F.} + \text{P.I.} = C_1 e^{-z} + C_2 e^{-2z} + \frac{1}{6}e^z$

$$\boxed{y = C_1 e^{\cos x} + C_2 e^{2\cos x} + \frac{1}{6}e^{-\cos x}}$$

---

**Ques** $x\frac{d^2y}{dx^2} + (4x^2 - 1)\frac{dy}{dx} + 4x^3 y = 2x^3$ [2013, 2016]

**Ans** Given equ$^n$ can be written as
$$\frac{d^2y}{dx^2} + \left(\frac{4x^2 - 1}{x}\right)\frac{dy}{dx} + 4x^2 y = 2x^2 \quad \text{—— (1)}$$

On comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = \frac{4x^2 - 1}{x}, \quad Q = 4x^2, \quad R = 2x^2$

Now choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$\Rightarrow \left(\frac{dz}{dx}\right) = 2x$

$\Rightarrow z = x^2$

$\Rightarrow \frac{d^2z}{dx^2} = 2$

Now,

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{——— (2)}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(\frac{4x^2-1}{x}\right)(2x)}{4x^2} \Rightarrow \boxed{P_1 = 2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} \Rightarrow \boxed{Q_1 = 1}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2x^2}{4x^2} \Rightarrow \boxed{R_1 = \frac{1}{2}}$$

Now from (2) $\quad \frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = \frac{1}{2}$

A.E $\quad m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

C.F $= (C_1 + zC_2)e^{-z}$

P.I $= \frac{1}{(D^2+2D+1)}\left(\frac{1}{2}\right) = \frac{1}{(D^2+2D+1)}\frac{1}{2}e^{0z} = \frac{1}{2}$

Hence $\quad y = C.F. + P.I. = (C_1 + zC_2)e^{-z} + \frac{1}{2}$





So $\quad \boxed{y = (C_1 + x^2C_2)e^{-x^2} + 1/2}$

**Ques** Solve $\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - y\sin^2 x = \cos x - \cos^3 x$ [2013]

**Ans** $\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - y\sin^2 x = \cos x - \cos^3 x$ ——— (1)

On comparing equ$^n$ (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Here

$P = -\cot x, \quad Q = -\sin^2 x, \quad R = \cos x - \cos^3 x$

Choose z such that

$\left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \frac{dz}{dx} = \sin x$

$\boxed{z = -\cos x}, \quad \frac{d^2z}{dx^2} = \cos x$

Now $\quad \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cot x(\sin x)}{\sin^2 x} = 0 \Rightarrow \boxed{P_1 = 0}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sin^2 x}{\sin^2 x} \Rightarrow \boxed{Q_1 = -1}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cos^3 x}{\sin^2 x} = \frac{\cos x(1-\cos^2 x)}{\sin^2 x} = \frac{\cos x \sin^2 x}{\sin^2 x}$$

$\boxed{R_1 = \cos x} \Rightarrow \boxed{R_1 = -z}$

So the differential equ$^n$ becomes

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$\frac{d^2y}{dz^2} - y = -z$ Auxiliary Equ$^n$ : $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$C.F. = c_1 e^{z} + c_2 e^{-z}$

$P.I. = \frac{1}{D^2 - 1}(-z) = (-1)(1 - D^2)^{-1}(-z)$

$= (-1)(1 + D^2)(-z) = z$

Hence $y = C.F. + P.I. = c_1 e^{z} + c_2 e^{-z} + z$

$\Rightarrow \boxed{y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x}$

H.W.

Ques $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2 y = \frac{1}{x^2}$ [2014]

Ans $y = c_1 \cos\left(\frac{a}{2x^2}\right) + c_2 \sin\left(\frac{a}{2x^2}\right) + \frac{1}{a^2 x^2}$

H.W.

Ques By changing the independent variable, solve the differential equation. [2015]

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2 y = x^4$$

Ans $y = c_1 \cos(x^2) + c_2 \sin(x^2) + \frac{x^2}{4}$





## Method of Variation of Parameters.

Consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{—— (1)}$$

Steps for solution (By variation of Parameters)

- Make the coefficient of $\frac{d^2y}{dx^2}$ as 1, if it is not so.
- Compare given differential equation with (1) and find R.
- Find out the part of C.F.
- Let $u$ and $v$ are part of C.F.
- Consider $y = Au + Bv$ be the complete sol$^n$

where $A = \int \frac{-Rv}{W} dx + c_1$ where $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

$B = \int \frac{Ru}{W} dx + c_2$ $\boxed{W = uv' - u'v}$

$\Rightarrow$ Hence $y = Au + Bv$ is complete solution.

$$\boxed{y = u\int \frac{-Rv}{W}dx + v\int \frac{Ru}{W}dx + c_1 u + c_2 v}$$

where R is function of $x$.

**Ques** Solve the following differential equation by variation of parameters.

$$\frac{d^2y}{dx^2} + a^2 y = \sec(ax) \quad [2013, 2014, 2015]$$

**Ans** Auxiliary Equn $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

C.F $= C_1 \cos(ax) + C_2 \sin(ax)$

Hence $\cos(ax)$ and $\sin(ax)$ are two part of C.F.

Let $u = \cos(ax)$ and $v = \sin(ax)$

Now $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$

Let $y = Au + Bv$ be the complete solution.

$A = \int \frac{-Rv}{W} dx + C_1$

$= \int \frac{-\sec(ax)\sin(ax)}{a} dx + C_1$

$= -\int \frac{\tan ax}{a} dx + C_1$

$A = \frac{1}{a^2} \log(\cos ax) + C_1$

$B = \int \frac{Ru}{W} dx + C_2$

$= \int \frac{\sec(ax)\cos(ax)}{a} dx + C_2$

$= \int \frac{1}{a} dx + C_2$

$B = \frac{x}{a} + C_2$

Hence Complete soln $y = Au + Bv$

$$y = \left[\frac{1}{a^2} \log(\cos ax) + C_1\right] \cos(ax) + \left[\frac{x}{a} + C_2\right] \sin(ax)$$

$$y = \frac{\cos(ax)}{a^2} \log(\cos ax) + \frac{x \sin(ax)}{a} + C_1 \cos(ax) + C_2 \sin(ax)$$





**Ques** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Solve it by using variation of parameter [2017]

**Ans** Auxiliary Equn $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$

Complementary function (C.F) $= (C_1 + xC_2) e^{3x}$

$= C_1 e^{3x} + C_2 (x e^{3x})$

Hence $e^{3x}$ and $xe^{3x}$ are two part of C.F.

Let $u = e^{3x}$ and $v = xe^{3x}$

Now $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix}$

$W = e^{6x} + 3xe^{6x} - 3xe^{6x} = e^{6x} \Rightarrow \boxed{W = e^{6x}}$

Let $y = Au + Bv$ be the complete soln

$A = \int \frac{-Rv}{W} dx + C_1$

$= \int \frac{-\frac{e^{3x}}{x^2} \cdot xe^{3x}}{e^{6x}} dx + C_1$

$= -\int \frac{dx}{x} + C_1$

$= -\ln x + C_1$

$B = \int \frac{Ru}{W} dx + C_2$

$= \int \frac{\frac{e^{3x}}{x^2} \cdot e^{3x}}{e^{6x}} dx + C_2$

$= \int \frac{dx}{x^2} + C_2$

$= -\frac{1}{x} + C_2$

Hence complete soln $y = Au + Bv$

$$y = (-\ln x + C_1) e^{3x} + \left(-\frac{1}{x} + C_2\right) x e^{3x}$$

$$y = -e^{3x} \ln x + C_1 e^{3x} - e^{3x} + C_2 x e^{3x}$$

$$y = (c_1 - 1)e^{3x} + c_2 x e^{3x} - e^{3x}\ln x$$

$$\boxed{y = c_1' e^{3x} + c_2 x e^{3x} - e^{3x}\ln x} \quad \left(\text{where } c_1' = c_1 - 1\right)$$

H.W. **Ques** $\frac{d^2y}{dx^2} + y = \tan x$ [2015]

**Ans** $y = c_1\cos x + c_2\sin x - \cos x \cdot \log_e(\sec x + \tan x)$

**Ques** Solve by the method of variation of parameter

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$$

**Ans** Here $e^x$ and $e^{2x}$ are part of C.F.

Let $u = e^x$, $v = e^{2x}$ then

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} \Rightarrow \boxed{W = e^{3x}}$$

Let $y = Au + Bv$ be the complete soln.

$$A = \int \frac{-Rv}{W}dx + c_1 = \int \frac{-\left(\frac{e^x}{1+e^x}\right)e^{2x}}{e^{3x}}dx + c_1$$

$$= -\int \frac{dx}{1+e^x} + c_1 = -\int \frac{e^{-x}dx}{(e^{-x}+1)} + c_1 = \ln(e^{-x}+1) + c_1$$

$$B = \int \frac{Ru}{W}dx + c_2 = \int \frac{\left(\frac{e^x}{1+e^x}\right)e^x}{e^{3x}}dx + c_2$$

$$= \int \frac{1}{e^x(1+e^x)}dx + c_2 = \int \frac{(1+e^x) - e^x}{e^x(1+e^x)}dx + c_2$$

$$= \int \frac{dx}{e^x} - \int \frac{dx}{1+e^x} + c_2$$





$$= \int e^{-x}dx - \int \frac{e^{-x}dx}{(e^{-x}+1)} + c_2$$

$$= -e^{-x} + \ln(e^{-x}+1) + c_2$$

Hence $y = Au + Bv$

$$y = [\ln(e^{-x}+1) + c_1]e^x + [\ln(e^{-x}+1) - e^{-x} + c_2]e^{2x}$$

$$\boxed{y = c_1 e^x + c_2 e^{2x} + e^x\ln(e^{-x}+1) + e^{2x}\ln(e^{-x}+1) - e^x}$$

**Ques** Use the variation of parameter method to solve the differential equation.

$$x^2y'' + xy' - y = x^2e^x \quad \text{—— (1)}$$

**Ans** Let $x = e^z \Rightarrow z = \ln x$

Let $\frac{d}{dz} = D$ and $\frac{d^2}{dz^2} = D(D-1)$

So from (1)

$$[D(D-1) + D - 1]y = e^{2z}e^{e^z}$$

$$\boxed{(D^2-1)y = e^{2z}e^{e^z}} \quad \text{—— (2)}$$

Auxiliary Equn $m^2 - 1 = 0 \Rightarrow m = \pm 1$

C.F. $= c_1e^{-z} + c_2e^z$

So $e^{-z}$ and $e^z$ are two part of C.F. of (2)

Let $u = e^{-z}$ and $v = e^z$ then

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-z} & e^z \\ -e^{-z} & e^z \end{vmatrix} = 2$$

Let $y = Au + Bv$ be the complete soln

$$A = \int \frac{-Rv}{W}\,dz + c_1 = \int \frac{-e^{2z}e^{e^z}e^{z}}{2}\,dz + c_1$$

$$= \frac{-1}{2}\int e^{2z}e^{e^z}e^{z}\,dz + c_1 \qquad \left(\begin{matrix}\text{Let } e^z = w \\ e^z dz = dw\end{matrix}\right)$$

$$= \frac{-1}{2}\int w^2 e^{w}\,dw + c_1$$

$$= \frac{-1}{2}\left[w^2e^{w} - 2(w-1)e^{w}\right] + c_2$$

$$= \frac{-1}{2}\left[e^{2z}e^{e^z} - 2(e^z-1)e^{e^z}\right] + c_2$$

$$B = \int \frac{Ru}{W}\,dz + c_2 = \int \frac{e^{2z}e^{e^z}e^{-z}}{2}\,dz + c_2$$

$$= \frac{1}{2}\int e^{z}e^{e^z}\,dz + c_2 \qquad \left(\begin{matrix}\text{Let } e^z = w \\ e^z dz = dw\end{matrix}\right)$$

$$= \frac{1}{2}\int e^{w}\,dw + c_2$$

$$= \frac{1}{2}e^{w} + c_2 = \frac{1}{2}e^{e^z} + c_2$$

Hence $y = Au + Bv$

$$y = \left\{-\frac{1}{2}\left[e^{2z}e^{e^z} - 2(e^z-1)e^{e^z}\right] + c_2\right\}e^{-z} + \left\{\frac{1}{2}e^{e^z} + c_2\right\}e^{z}$$

$$y = c_1e^{-z} + c_2e^{z} + (1 - e^{-z})e^{e^z}$$

$$\boxed{y = c_1\left(\frac{1}{x}\right) + c_2x + \left(1 - \frac{1}{x}\right)e^{x}}$$ Ans





Complementary functions.

| | Roots of auxiliary equn | Corresponding complementary fun |
|---|---|---|
| 1 | One real root $m_1$ | $c_1e^{m_1x}$ |
| | Two real and different roots | $c_1e^{m_1x} + c_2e^{m_2x}$ |
| | Three real and different roots | $c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}$ |
| 2 | Two real and equal roots | $(c_1 + xc_2)e^{m_1x}$ |
| | Three real and equal roots | $(c_1 + xc_2 + x^2c_3)e^{m_1x}$ |
| 3 | One pair of complex roots $(\alpha \pm i\beta)$ | $e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$ |
| | Two pairs of complex and equal root | $e^{\alpha x}[(c_1 + xc_2)\cos\beta x + (c_3 + xc_4)\sin\beta x]$ |
| 4 | One pair of surd roots $(\alpha \pm \sqrt{\beta})$ | $e^{\alpha x}[c_1\cosh(\sqrt{\beta}x) + c_2\sinh(\sqrt{\beta}x)]$ |
| | Two pairs of surd and equal roots $\alpha \pm \sqrt{\beta}$, $\alpha \pm \sqrt{\beta}$ | $e^{\alpha x}[(c_1 + xc_2)\cosh(\sqrt{\beta}x) + (c_3 + xc_4)\sinh(\sqrt{\beta}x)]$ |